INDEX TO ANSWERS

ANSWERS FROM ISSUE 60

SOME TRIANGLE NUMBERS – 2

Many thanks to Andrew Palfreyman who found ***five***, not four solutions!

	7	
1	0	5
	3	

Grid A

	7	
3	0	0
	3	

	7	
4	0	6
	3	

	7	
9	0	3
	3	

	7	
9	4	6
	1	

	6		6
1	3	2	6
2	0	1	6
0		0	

Grid B

CROSSNUMBER

Many thanks again to Andrew Palfreyman who pointed out that 1 Down and 8 Across do not give unique answers so there are four possible solutions.

[1] 1	4	4		[2] 8	
3			[3] 9	1	9
[4] 2	3	[5] 1	0		
		[6] 1	0	9	[7] 8
[8] 3	[9] 6	1			0
	5		[10] 3	4	3

[1] 1	4	4		[2] 8	
8			[3] 9	1	9
[4] 2	3	[5] 1	0		
		[6] 1	0	9	[7] 8
[8] 3	[9] 6	1			0
	5		[10] 3	4	3

[1] 1	4	4		[2] 8	
3			[3] 9	1	9
[4] 2	3	[5] 1	0		
		[6] 1	0	9	[7] 8
[8] 9	[9] 6	1			0
	5		[10] 3	4	3

[1] 1	4	4		[2] 8	
8			[3] 9	1	9
[4] 2	3	[5] 1	0		
		[6] 1	0	9	[7] 8
[8] 9	[9] 6	1			0
	5		[10] 3	4	3

TREASURE HUNTS 12, 13

12
This is a rostral column in St Petersburg, Russia. Built in 1811 it follows the example of the Romans who used the prows (rostrum) of conquered ships to decorate their monuments.

13
The school with the plaque shown is Trinity-House School in Newcastle. It came into being in 1505 and teachers included the renowned mathematician Robert "Beau" Harrison in 1756 and John Fryer, the land surveyor famed for his maps of both Newcastle and Northumberland, in 1771. Edward Riddle was Schoolmaster between 1814 and 1821, when he was appointed Master of the Royal Naval College, Greenwich. There is more about other mathematics teachers at http://www.british-history.ac.uk/no-series/newcastle-historical-account/pp443-445

SYMKEN - 5

Easy

2x 1	5 5	24x 3	2	4
2	1	160x 4	15x 3	5
15x 3	2	5	4	1
5	48x 4	2x 2	1	6x 3
4	3	6+ 1	5	2

Moderate

6+ 5	7+ 3	9+ 4	3+ 2	1
1	4	5	5+ 3	2
4+ 3	1	7+ 2	5	12+ 4
11+ 2	5	9+ 1	4	3
4	2 2	3	1	5

NOORTS AND CROXXES - 2

o	o	x	o	x	o	x	x	o	x
x	o	x	o	o	x	o	x	x	o
o	x	o	x	x	o	x	o	o	x
x	o	x	x	o o	o	x	o	x	o
x	x	o	o	x	x	o	x	o	o
o	o	x	x	o	x	o	o	x	x
x	x	o	x	o	o	x	x	o	o
o	o	x	o	x	x	o	x	x	o
o	x	o	o	x	x	o	o	x	x
x	x	o	x	o	o	x	o	o	x

o	x	x	o	x	o	o	x	x	o
o	x	x	o	o	x	o	o	x	x
x	o	o	x	o	x	x	o	o	x
x	x	o	x	x	o	o	x	o	o
o	o	x	o	x	o	x	o	x	x
x	o	o	x	o	x	x	o	x	o
x	x	o	o	x	x	o	x	o	o
o	o	x	x	o	o	x	x	o	x
x	x	o	x	o	o	x	o	x	o
o	o	x	o	x	x	o	x	o	x

o	o	x	x	o	o	x	x	o	o	x	x	o	x
x	o	x	o	o	x	x	o	x	x	o	o	x	o
o	x	o	x	x	o	o	x	x	o	o	x	o	x
o	o	x	x	o	x	x	o	o	x	x	o	o	x
x	x	o	o	x	x	o	o	x	x	o	o	x	o
x	x	o	o	x	o	x	x	o	o	x	x	o	o
o	o	x	x	o	o	x	x	o	o	x	o	x	x
x	x	o	o	x	x	o	o	x	x	o	x	o	o
x	x	o	o	x	o	o	x	o	x	o	x	o	x
o	o	x	x	o	x	x	o	x	o	x	o	x	o
o	x	x	o	o	x	o	x	x	o	o	x	x	o
x	o	o	x	x	o	o	x	o	x	x	o	o	x
o	o	x	x	o	o	x	o	o	x	x	o	x	x
x	x	o	o	x	x	o	o	x	o	o	x	x	o

ANSWERS FROM ISSUE 61

NEW KID ON THE BLOCK

One solution is {1, 2, 3, 5, 6, 7, 8, 11, 12, 14, 15, 20, 21, 24, 28, 30, 34, 37, 38, 40} and

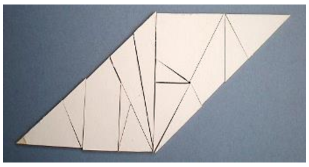
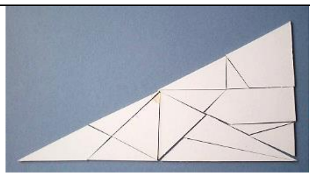
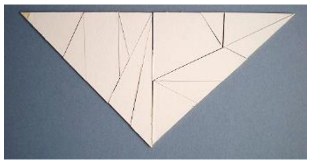
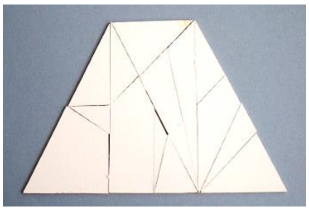
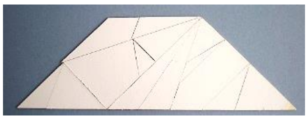
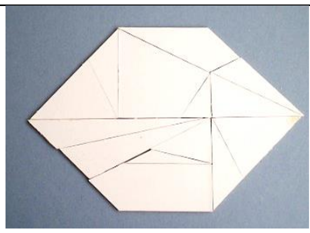
{4, 9, 10, 13, 16, 17, 18, 19, 22, 23, 25, 26, 27, 29, 31, 32, 33, 35, 36, 39}

A 2-DIMENSIONAL PROBLEM

Area of $EFGH = 90\text{cm}^2$.

If P divides EH in the ratio $1{:}x$ then the area of the large square is $\dfrac{10(2x-1)^2}{(x-1)^2}$.

STOMACHION

parallelogram	rectangle	right-angled triangle	right-angled isosceles triangle
			
trapezium	different trapezium	hexagon	
			

PLACING NUMBERS

9	4	1	5	2	6		3		5	
0		3		5		4	7	1	8	0
8	2	7	9	4	1	9	2		2	
8	1	3	4	6		6	9	6	4	7
7	2	8	7		3	8	0	1	5	
	9		8	7	1	4		9		8
6	1	5	0		4		7	0	2	4
7		3		3	2	5	1	3		3
5	0	4	5	9	8		9	0	3	6
6		9		6	5	7	7	8	5	0
4	8	8	4	3	7		8		7	
9		3		6		2	5	1	6	0
4	6	2	7	1	8	9	0		6	
	7	5	6	2	5	0		9		1
5	8	2	1		5	6	6	2	3	4
	2		3	4	4	3		2		1
3	0	1	0		2	9	8	1	7	0

QUICKIE 30

If the sides of the rectangle are x and y with $x < y$, then $4y + 5x = 264$ and $3x = 2y$.
This gives $x = 24$ and $y = 36$, so the perimeter is 120cm and the area is 864cm^2.

QUICKIE 31

$$1014492753623188405797 \times 7 = 7101449275362318840579$$

SYMKEN - 6

Easy

5x 5	12x 2	3	6÷ 1	6 6	12x 4
1	72x 4	2	6	11+ 5	3
3	6	12x 4	2 2	1	5
48x 4	3	1	5 5	12x 2	6
6	5x 1	30x 5	12x 3	8x 4	2
2	5	6	4	4+ 3	1

Moderate

9+ 3	4- 2	6	5+ 4	5÷ 5	1
6	12+ 3	5	1	2÷ 2	4
6÷ 1	6	4	10+ 5	3	2
3+ 2	1	4+ 3	10+ 6	4	8+ 5
4 4	5 5	1	3÷ 2	6	3
9+ 5	4	1- 2	3	5- 1	6

MALCOLM AND HIS FLIGHTS

The three consecutive odd numbers are 53, 55 and 57. The total is 165 minutes.

PI PIX

We have $\pi r^2 = 2r + \pi r$ leading to $r = \dfrac{2+\pi}{\pi}$.

TREASURE HUNTS 14, 15

TH 14 was York railway station, designed by Thomas Prosser and William Peachey. It was opened in 1877. The girder indicated by a yellow line appears to be part of an ellipse.
TH 15 is the grave of Florence Nightingale in the churchyard of St Margaret of Antioch, Wellow, Hampshire
Adam Mallis was the first name drawn out of the hat in each case and so won the prizes.

ANSWERS FROM ISSUE 62

DSL HZMT GSRH HLMT? (WHO SANG THIS SONG?)

Queen sang it: it is the opening words from Bohemian Rhapsody.

Is this the real life?
Is this just fantasy?
Caught in a landslide
No escape from reality
Open your eyes
Look up to the skies and see
I'm just a poor boy, I need no sympathy
Because I'm easy come, easy go

The letter that appears in position n in the alphabet has been replaced by the letter at position $27 - n$ i.e. the alphabet has been reversed.

CROSSWORD

1 D	E	2 C	I	3 M	A	4 L		5 T	O	W	E	6 R
A		W		E		A		H				H
7 I	N	T	E	G	E	R		8 R	A	9 T	I	O
L				A		G		E		O		M
10 Y	A	R	11 D			12 E	13 V	E	N	T		B
			I				A			A		U
14 S	I	15 D	E		16 N	E	T		17 P	L	U	S
P		A			O				E			
H		18 T	E	19 N	T	20 H			21 G	I	L	22 L
E		U		I		O		23 R				I
24 R	O	M	A	N		25 U	N	I	F	26 O	R	M
E				T		R		N		N		I
27 S	O	U	T	H		28 S	E	G	M	E	N	T

SYMKEN – 7

Easy

24x 2	4	15x 3	1	5 5
3	6x 1	5	2 2	8+ 4
9+ 5	2	4 4	3	1
4	3	10x 1	5	1- 2
4- 1	5	2	4 4	3

Moderate

36x 3	3+ 1	32x 4	2	15x 5
1	2	50x 5	4	3
4	3	2	5	1
7+ 2	15x 5	3	10+ 1	4
5	4 4	1 1	3	2

SCORES IN THREE SUBJECTS

By thinking systematically, and creating tables like those below, where each row represents one or more students with identical sets of scores across the subjects, you can be sure that you have accounted for all possible scenarios.

With at least 4 students, we have 2 possibilities that are consistent with the first two given graphs. These correspond to the first two of the six graphs shown above, and are numbered accordingly.

1

Art	Geography	Chemistry
1	3	1
2	1	1
2	3	2
3	2	3

2

Art	Geography	Chemistry
1	3	2
2	1	1
2	3	1
3	2	3

With at least 5 students, we have 4 more possibilities that are consistent with the first two given graphs. These correspond to the last four of the six graphs shown above, and are numbered accordingly.

3

Art	Geography	Chemistry
1	3	1
1	3	2
2	1	1
2	3	1
3	2	3

4

Art	Geography	Chemistry
1	3	1
1	3	2
2	1	1
2	3	2
3	2	3

5

Art	Geography	Chemistry
1	3	1
2	1	1
2	3	1
2	3	2
3	2	3

6

Art	Geography	Chemistry
1	3	2
2	1	1
2	3	1
2	3	2
3	2	3

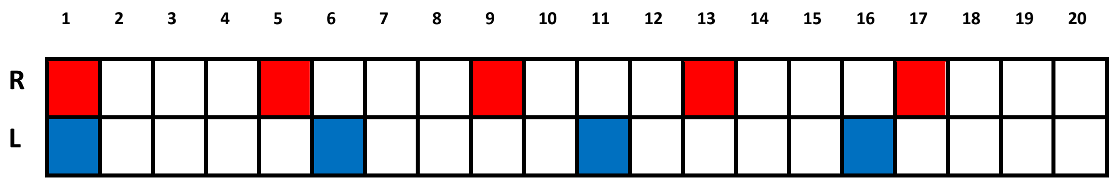
So overall there are 6 possibilities, leading to the 6 graphs shown.

RHYTHM AND POLYRHYTHM

1 Phase length is 20

2 20 is the least common multiple of 5 and 4

3



SYMdoku 2016-2017

2	**0**	**1**	**6**	4	3	7	5
7	4	3	5	0	1	6	2
0	1	2	7	5	4	3	6
3	6	5	4	1	7	2	0
6	3	0	2	7	5	4	1
5	7	4	1	6	2	0	3
1	2	7	0	3	6	5	4
4	5	6	3	**2**	**0**	**1**	**7**

QUICKIE 32

$105263157894736842 \times 2 = 210526315789473684$

TREASURE HUNTS 16, 17

TH16 This is the Garabit Viaduct, a railway arch bridge spanning the River Truyère near Ruynes-en-Margeride, Cantal, France. The bridge was constructed between 1882 and 1884 by Gustave Eiffel.

TH17 18 x 3 x 8 x 8 = 3456 plants in the red trapezium.
No entries were received for any of these treasure hunts.

ANSWERS FROM ISSUE 63

SYMKEN – 8

Very easy

6+ 4	5+ 3	2	2- 1
2	4÷ 1	4	3
1 1	1- 4	3	2- 2
6x 3	2	1 1	4

Easy

6+ 4	3- 1	1- 2	3
2	4	36x 3	8X 1
3÷ 1	3	4	2
3	1- 2	1	4

Quite easy

3+ 2	2- 3	1	7+ 4
1	8+ 4	16x 2	3
3	1	4	2
2- 4	2	2- 3	1

Moderate

2- 1	3	2÷ 4	2
24x 4	1 1	5+ 2	3
3	2	1 1	3- 4
2 2	7+ 4	3	1

TWO SPECIAL FRAMES

Q1 The area of a trapezium is the sum of the parallel edge lengths, divided by 2, multiplied by the width.
Join corresponding points on each shape to get a set of trapezia.

To sum their areas, you need not do a separate calculation for each, you can just add all the edge lengths, divide by 2 and multiply by the width.

But you know from what you are given that the edge length total is the same for A and B.
And you are also given that the blue area is the same for A and B.

Therefore their widths must be the same.

Q2 Equating areas: $(k^2-1)s^2=\frac{3\times 4}{2}(l^2-1)$. **(I)**

Equating perimeters: $4(k+1)s=(3+4+5)(l+1)$. **(II)**

Using the width information: $\frac{(k-1)s}{2}=2$. **(III)**

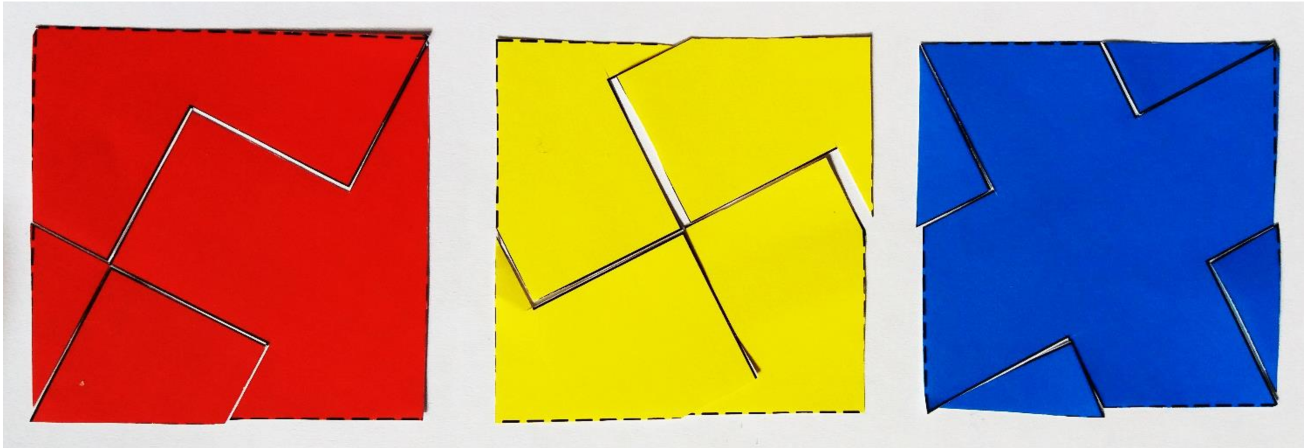
Dividing (**I**) by (**II**) and substituting for *s* from (**III**), you find $l = 3$. This gives you the required dimensions.

If you do not believe the answer to Q1, you can check by substituting back in (**I**) to find the area, and equating to B's area written as the sum of its trapezia, i.e. the perimeter total $\times\frac{w}{2}$.

AREA AND PERIMETER PROBLEM

Small triangle: area $\frac{1}{12}$ perimeter $\frac{5+\sqrt{13}}{6}$. Large triangle: area $\frac{1}{3}$ perimeter $\frac{5+\sqrt{13}}{3}$.

SQUARES FROM CROSSES



Now you can make a rectangle by putting the three squares together.

Use Pythagoras' Theorem to work out the perimeter of each piece by concentrating on the dotted lines.

QUICKIE 33

$105264 \times 4 = 410526$

ANSWERS FROM ISSUE 64

SYMKEN – 9

Easy

2- 4	1 1	1- 2	3
2	1- 3	5+ 1	4 4
2- 3	2	4	2x 1
1	1- 4	3	2

Quite easy

1- 1	2- 3	10+ 4	2
2	1	1- 3	4
24x 3	4	2	1 1
4 4	2	2- 1	3

Moderate

3- 4	1	1- 2	2- 3
2- 2	4	3	1
7+ 1	3	2- 4	2
3	8x 2	1	4

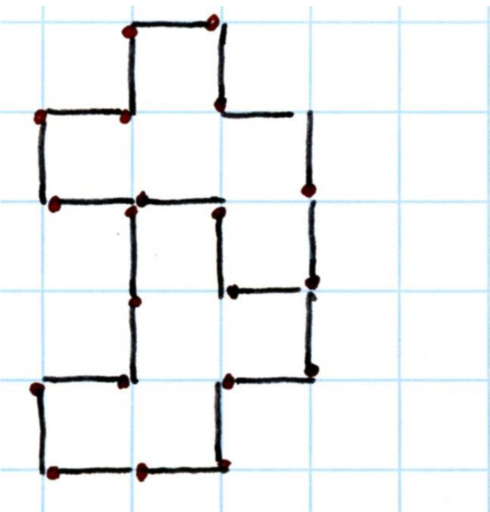
Moderate

5+ 3	1	8+ 2	4
1	1- 3	4	2
32x 4	1- 2	3	3x 1
2	4	1	3

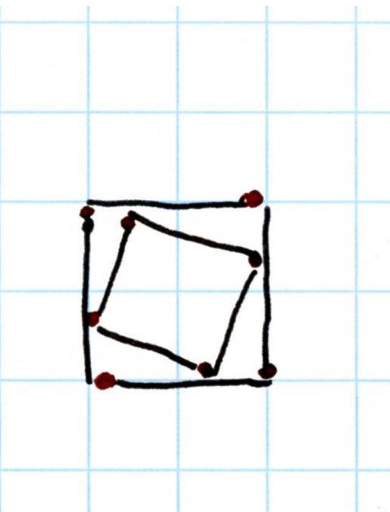
MORE MATCHSTICK PUZZLES

There's always one that slips through the net! Unfortunately Puzzles 2 and 3 were not meant to be the same and I missed noticing that until I came to do the answers.

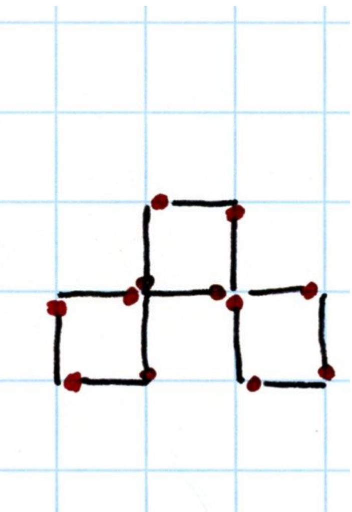
Puzzle 1



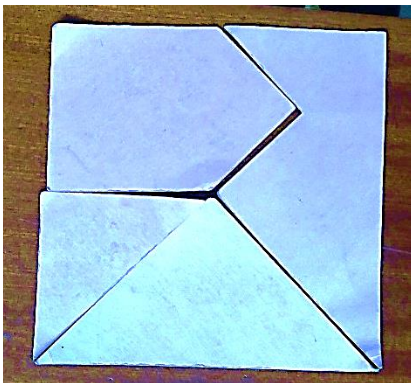
Puzzles 2 and 3



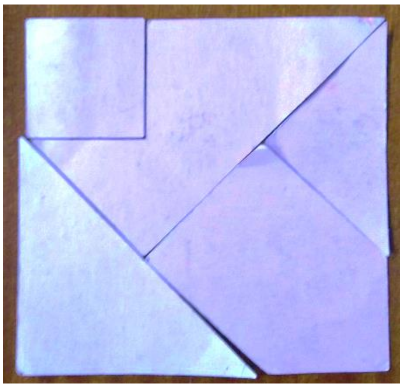
Puzzle 4



A DOUBLE DISSECTION



Area 16 squares



Area 18 squares

MUYB R'A HRNJP-FCKG **WHEN I'M SIXTY-FOUR**

When I get older losing my hair
Many years from now
Will you still be sending me a valentine
Birthday greetings, bottle of wine?
If I'd been out till quarter to three
Would you lock the door?
Will you still need me, will you still feed me
When I'm sixty-four?

The coding used was as follows:

A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
S	I	X	T	Y	F	O	U	R	V	W	Z	A	B	C	D	E	G	H	J	K	L	M	N	P	Q

The alphabet is written out and then SIXTYFOUR, the codeword, is written below the first letters of the alphabet, followed by the missing letters in alphabetical order from the last letter of SIXTYFOUR.

The message is then coded, so the first letter of the message, W, becomes M etc.

TEST YOUR DIVISION!

$498 \div \boxed{3} = 166$ $682 \div \boxed{22} = 31$ $952 \div \boxed{7} = 136$ $2555 \div \boxed{5} = 511$ $6669 \div \boxed{19} = 351$ $56144 \div \boxed{8} = 7018$

QUICKIE 34

$1016949152542372881355932203389830508474576271186440677966 \times 6 =$
$6101694915254237288135593220338983050847457627118644067796$

Of course, $1016949152542372881355932203389830508474576271186440677966 \times (10^{58} + 1)$ multiplied by 6 also works. In fact, there are an infinite amount of numbers found by placing $1016949152542372881355932203389830508474576271186440677966$ in front of each of each solution you find.

ANSWERS FROM ISSUE 65

TWIN PRIMES

1997 and 1999 are the 61st prime pair

TREASURE HUNT 18

The speed limit represents 14 kmh^{-1}. Its location is at Tortworth Court, Wotton Under Edge, Gloucester, GL12 8HH.

CROSSNUMBER

9	6	1		2	
9			3	7	7
1	0	2	4		
		6	5	6	1
1	4	3			3
	9		4	2	9

TRIANGLE SEARCH

1 12 triangles **2** 15 triangles **3** $3n + 3$ triangles **4** 33 triangles **5** 24 triangles

6 30 triangles **7** $6n + 6$ triangles **8** 66 triangles **9** 40 triangles **10** 50 triangles

11 $10n + 10$ triangles **12** 110 triangles **13** 75 triangles **14** Your own explanation **15** $\frac{(n+1)(n+2)(m+1)}{2}$

		Number of internal lines from top vertex				
Number of internal parallel lines		0	1	2	3	4
	0	1	3	6	10	15
	1	2	6	12	20	30
	2	3	9	18	30	45
	3	4	12	24	40	60
	4	5	15	30	50	75
	n	$n+1$	$3n+3$	$6n+6$	$10n+10$	$15n+15$

PUZZLE PAGE

Easy

2	1	2	4
3	4	3	1
5	1	5	2
2	4	3	1
3	1	5	2

Medium

1	2	3	4
3	4	1	2
1	2	5	4
5	4	3	2
3	1	5	1

A GOOD AGE

Aneeta was born in 2008.

TEST YOUR MULTIPLICATION

There are many solutions. Here are three: $4093 \times 7 = 28651$, $5694 \times 3 = 17082$, $7039 \times 4 = 28156$.

NOTHING 2 IT 520 zeroes if you include 2018

Harder

1	2	1	3	1	3	5	1	2
5	4	5	4	5	2	4	3	4
1	3	2	3	1	3	1	2	1
5	4	1	5	2	4	5	4	3
2	3	2	4	3	1	2	1	2

Very hard

1	2	4	2	4	1	4	1	4
4	5	1	5	3	5	3	5	3
1	2	3	2	1	2	1	2	1
3	5	1	4	3	4	3	5	4
1	2	3	2	5	1	2	1	2

CLUES FROM VARIOUS CROSSWORDS

1 Rule of three **2** Quadrilateral **3** Thousand **4** Euler **5** Algebra

6 Acute-angle **7** Turin **8** Its Greek to me **9** Archimedean **10** Threescore

11 Tetrahedra **12** Eleven **13** Mathematician

DAVOOD'S DICE

4

PYRAMID NETS

None of them are nets of a pyramid. Net A is totally flat, some of the sides of net D that meet are of different lengths, and although you can pair up equal length sides in nets B and C, you cannot get them all to meet at a vertex.

QUICKIE 36 – Pecking chickens

36 – Note that for a chicken not to be pecked, the chicken on the right must peck to the right and the chicken on the left must peck to the left, so the probability a chicken is not pecked is $\frac{1}{2}\times\frac{1}{2}=\frac{1}{4}$. So, the expected number of unpecked chickens is $\frac{1}{4}\times 144=36$. If the chickens are in a straight line, then the chicken at each end will be unpecked with probability $\frac{1}{2}$. So the expected number of unpecked chickens is $2\times\frac{1}{2}+142\times\frac{1}{4}=36\frac{1}{2}$.

Eagle-eyed readers might have noticed that this should have been Quickie 35 – oops!

ARRAS GIANTS

Taking the height of the human as 180 cm and mass as 70 kg, then L'ami Bidasse is about 5 metres high and would have a mass of about 1.5 tonnes. Dédé would be about 4 metres tall and have a mass of about 850 kg.

ANSWERS FROM ISSUE 66

MAGIC SQUARES WITH A DIFFERENCE

24	3	18
9	15	21
12	27	6

2	-3	4
3	1	-1
-2	5	0

2	17	11
19	10	1
9	3	18

7	2	3
0	4	8
5	6	1

RED TRIANGLE

The amazing result is that the area is always $\frac{1}{2}x^2$, no matter what the size of the squares are.

NUMBER POLITENESS

Jonathan Tolan let me know on August 2 that the definition of polite numbers should have been defined more carefully! They should be defined as numbers that are the sum of two or more consecutive positive integers. I had let that slip slightly as it had been left as being two or more consecutive whole numbers and whole numbers include 0, which is not a positive number. Some sources include 0 as a natural number, some do not so there is no agreement on the set of natural numbers.

1. 1, 2, 4, 8
2. The sequence continues 16, 32, 64, … The impolite numbers are exactly powers of 2, i.e. 2^n.
3. 6
4. 10
5. 15
6. 5050, this is the sum of all numbers from 1 to 100.
   The sequence of numbers which are sums of consecutive integers are referred to as triangular.
   The expression for triangular numbers is $n(n + 1)/2$.
7. 18 = 5 + 6 + 7 = 3 + 4 + 5 + 6
8. There was no question here!
9. One way of calculating the politeness of a positive number is to decompose the number into its prime factors, taking the powers of all the prime factors greater than 2, adding 1 to all of the powers, multiplying the numbers thus obtained and subtracting 1.
   e.g. $15 = 3^1 \times 5^1$.
   (1 + 1) (1 + 1) – 1 = 3
   Therefore, the number of ways of expressing 15 as the sum of two or more consecutive natural numbers is 3.
   Here they are: 15 = 1 + 2 + 3 + 4 + 5 = 4 + 5 + 6 = 7 + 8.
10. 30 has a politeness of 3, 75 a politeness of 5, 70 a politeness of 3, 135 a politeness of 7 and 8192 is impolite as it is a power of 2.

Polite Friends

$42=3+4+5+6+7+8+9=9+10+11+12$
$65=2+3+4+5+6+7+8+9+10+11=11+12+13+14+15$
$70=12+13+14+15+16=16+17+18+19$
$99=4+5+6+7+8+9+10+11+12+13+14=14+15+16+17+18+19$

and continues with 105, 117, 133, 135, 154, 175, 180.

Polite Neighbours

$30=4+5+6+7+8=9+10+11$
$42=9+10+11+12=13+14+15$
$75=3+4+5+6+7+8+9+10+11+12=13+14+15+16+17$
$90=16+17+18+19+20=21+22+23+24$

and continues with 105, 135, 147, 165, …

Interestingly, 30, 42, 105, 135 are both polite friends and polite neighbours!

FIVE CARD MAGIC TRICK

The hidden cards are the Jack of Clubs, the 7 of clubs and the King of Hearts.

PUZZLES

1. The largest sum is 981 = 736 + 245. Wil Ransome, the editor of Mathematical Pie, informed me on June 22 that you can also have 981 = 627 + 354. Of course, there are other answers (by switching the digits in the same position (HTU), but not between positions) that give 981, but that is the largest sum. The smallest difference is 124 = 783 – 659.

2. 48 right-angled triangles. From each vertex there are three ways using two edges and three ways using an edge and a face. (Eagle-eyes will have noticed that there are two questions numbered 1 and no question 2.)

3. 48 people

4. Here are examples where T = 20 and where T = 18 (which is the minimum possible value for T.)

10	5	2	3
6	T = 20		9
4	7	1	8

8	3	5	2
9	T = 18		10
1	4	7	6

5. Andrew Palfryman informed me on July 12 that there was a mistake in the following answer.

There are 22 possibilities. If the stamps are labelled ABC on the top row and DEF on the bottom row, then if 3 stamps remain they can be ABD, ABE, BCE, BCF, DEA, DEB, EFB, EFC. If four stamps remain they can be ABCD, ABCE, ABCF, DEFA, DEFB, DEFC, ABEF, BCDE. If five stamps remain they can be ABCDE, ABCDF, ABCEF, ABDEF, ACDEF, BCDEF. This problem was posed by Brian Stokes in *The New Zealand Mathematics Magazine* of November 2008.

The question mentioned that two or more stamps were torn off and the above solution includes just one stamp torn off (which is the case if five remain). Therefore, the correct answer to the question as set is that there are just 16 possibilities.

6. $123+456+78+9=666=9+87+6+543+21$

ANSWERS FROM ISSUE 67

GRAECO-LATIN SQUARES

Some solutions are given here: they are not unique.

Part 1

1	2	3	4
4	3	2	1
2	1	4	3
3	4	1	2

A	B	C	D
C	D	A	B
D	C	B	A
B	A	D	C

A1	B2	C3	D4
C4	D3	A2	B1
D2	C1	B4	A3
B3	A4	D1	C2

Part 2

1	2	3	4	5
2	3	4	5	1
3	4	5	1	2
4	5	1	2	3
5	1	2	3	4

A	D	B	E	C
B	E	C	A	D
C	A	D	B	E
D	B	E	C	A
E	C	A	D	B

A1	B4	C2	D5	E3
B2	C5	D3	E1	A4
C3	D1	E4	A2	B5
D4	E2	A5	B3	C1
E5	A3	B1	C4	D2

Part 3

In the 18th century, Leonard Euler proved that it was possible to make Graeco-Latin squares of order n for any odd number or any number that was a multiple of 4. However, the other even numbers 2, 6, 10, 14 would not cooperate and the corresponding Graeco-Latin squares remained elusive.

He suggested that squares of the form $4n + 2$ could not be done. These numbers are known troublemakers – they are the only ones that are *not* the difference of two squares.

In 1959 some counterexamples to Euler's conjecture were discovered. An example of a square of the order 10 was published in the *New York Times* and caused renewed interest.

In the same year, Bose, Shrikhande and Parker showed that Euler was on this occasion wrong - there are Graeco-Latin squares of any order greater than 2 except for $n = 6$.

BAB'S BRICKS

Least volume of a brick is $2.5\times 3.5\times 4.5 = 39.375\,\text{cm}^3$, greatest volume is $3.5\times 4.5\times 5.5 = 86.625$86.625 cm^3.

To find out how many she could definitely store you need to work with the maximum size brick and the minimum internal dimensions, trying to pack in the best possible way. This turns out to be as follows:

$45.5\div 4.5 = 10.11\ldots$, $44.5\div 5.5 = 8.09\ldots$, $35.5\div 3.5 = 10.14\ldots$, so giving $10\times 8\times 10 = 800$ bricks.

To find out how many she could possibly store you need to work with the minimum size brick and the maximum internal dimensions, trying to pack in the best possible way. This turns out to be either of the following:

$45.5\div 2.5 = 18.2\ldots$, $36.5\div 4.5 = 8.11\ldots$, $46.5\div 3.5 = 13.2\ldots$, so giving $18\times 8\times 13 = 1872$ bricks, or

$45.5\div 3.5 = 13$, $36.5\div 4.5 = 8.11\ldots$, $46.5\div 2.5 = 18.6\ldots$, also giving $13\times 8\times 18 = 1872$ bricks.

STOP THE CLOCK!

If a diners ate between 6pm and 7 pm, and the increase each hour was d more diners, then the number of diners each hour was:

Time	*Cost (£)*	*Number of diners*
6 pm – 7 pm	6	a
7 pm – 8 pm	7	$a + d$
8 pm – 9 pm	8	$a + 2d$
9 pm – 10 pm	9	$a + 3d$
10 pm – 11 pm	10	$a + 4d$
11 pm – 12 am	11	$a + 5d$
12 am – 1 am	12	$a + 6d$

So the mean cost of a meal is $\frac{6a+7(a+d)+8(a+2d)+9(a+3d)+10(a+4d)+11(a+5d)+12(a+6d)}{a+(a+d)+(a+2d)+(a+3d)+(a+4d)+(a+5d)+(a+6d)} = \frac{63a+217d}{7a+21d}$,

and this must be equal to 9.96. A bit of algebra shows that this simplifies to $\frac{d}{a} = \frac{6}{7}$, or $a : d = 7 : 6$, giving solutions such as $a = 7$, $d = 6$ or $a = 14$, $d = 12$, etc.

Checking, with $a = 7$ and $d = 6$, we find a total spend of £1743 with 175 diners, giving a mean of £9.96.

I received two comments about this issue, concerning the mathematics in it, so have included these here, under the appropriate titles.

AN INCLINED PLANE IN THE OLD QUAY AT NANTES, FRANCE

Graham Hoare quite rightly pointed out the statement in the article 'If the angle of the slope is α, then the acceleration down the slope is $g\sin\alpha$', is questionable. He is quite correct. He goes on to say 'If the block of mass m slides down such a plane then the acceleration is $g(\sin\alpha - \mu\cos\alpha)$, m being the coefficient of friction. For a sphere, rolling without sliding, down a similar slope, the acceleration is $\frac{5}{7}g\sin\alpha$ and for a cylinder the corresponding result is $\frac{2}{3}g\sin\alpha$. In the real-life situation there are, as the note indicates, various constraints; the examples mentioned above represent ideal cases. Perhaps it would be more accurate to say that the force due to the weight down the plane is reduced by a factor $\sin\alpha$.'

Some reflections on Symmetry-Plus No.67 Autumn 2018 – comments from Bob Burn

Graeco-Latin Squares

These are the 25 points of a plane using coordinates modulo 5.

(0, 4)	(1, 4)	(2, 4)	(3, 4)	(4, 4)
(0, 3)	(1, 3)	(2, 3)	(3, 3)	(4, 3)
(0, 2)	(1, 2)	(2, 2)	(3, 2)	(4, 2)
(0, 1)	(1, 1)	(2, 1)	(3, 1)	(4, 1)
(0, 0)	(1, 0)	(2, 0)	(3, 0)	(4, 0)

Lines of the form $y = mx + c$ and $y = mx + d$ are parallel, so if we choose a particular gradient m and then enter the value of c for the line $y = mx + c$ through each point, we get a Latin square, for that value of m. Each table below gives the "c" value for the line through each point, working modulo 5. The different squares below are for different values of m: 1,2, 3 and 4. If one square is Latin and another one is Greek, they may be superimposed to give a Graeco-Latin square.

4	3	2	1	0
3	2	1	0	4
2	1	0	4	3
1	0	4	3	2
0	4	3	2	1

$m = 1$

4	2	0	3	1
3	1	4	2	0
2	0	3	1	4
1	4	2	0	3
0	3	1	4	2

$m = 2$

4	1	3	0	2
3	0	2	4	1
2	4	1	3	0
1	3	0	2	4
0	2	4	1	3

$m = 3$

4	0	1	2	3
3	4	0	1	2
2	3	4	0	1
1	2	3	4	0
0	1	2	3	4

$m = 4$

This idea also works modulo 3 or 7 (and indeed for any prime). The four by four Graeco-Latin squares can be developed this way using the field $\{0, 1, a, a^2\}$ with $a^2 = a + 1$ (modulo 2). And for eight by eight Graeco-Latin squares using the field
$\{0, 1, b, b + 1, b^2, b^2 + 1, b^2 + b, b^2 + b + 1\}$ with $b^3 = b + 1$ (modulo 2).

The non-existence proof for six by six is really hard.

Take your seat

Since the seats come in rows of four (two each side) one might expect the difference between similar seats to be a multiple of 4.

Georg Cantor

There is mention here of both Dedekind's and Cantor's construction of the real numbers.

Each of these methods faced up to two unproved claims in Cauchy's 1821 exposition of Algebraic analysis. Dedekind worked on the proposition that an increasing sequence of rationals with an upper bound has a limit (separating the reached from the unreached rationals), while Cantor worked on the proposition that a sequence which is eventually as close as you like to its own terms has a limit. Cantor's method was refined in the 20th century using equivalence classes of such sequences to define a real number.

Cantor's flight of imagination

At the beginning of Cantor's paper of 1891 in which he gives the diagonal argument for uncountability, he starts with two different things a and b. He makes lists, using just these two symbols, and asks whether a full collection of such lists, might be countable, that is whether such a list can be matched with $1, 2, 3, \ldots$. This approach to diagonalisation is not troubled by suffices. [Rather incidentally, Archimedes did use infinitesimals in "The Method", but in his Introduction he said that these did not provide proofs.]

ANSWERS FROM ISSUE 68

CROSSNUMBER

1	4	7		9	7
2		2	1	0	
4		9		1	3
8	1		3		2
	4	8	4		3
2	4		5	1	2

CRACK THE CODE (CLUE: 2)

I included this since I reckon the song has a link with probability! (and I like ABBA songs)
It uses what is known as a 'rail fence cipher', in this case with two rails.
Each line was written on two rails: e.g.

E		C		L		N		W		S		R		T		E		O		T		O		A		L	
	A		H		I		E		A		W		I		T		N		N		W		R		I		S

Then the letters are written as one line, without spaces: ECLNWSRTEOTOALAHIEAWITNNWRIS
To decipher the message, with the clue there are two rails, split the line into two
ECLNWSRTEOTOAL AHIEAWITNNWRIS
and write as a block with two rows
ECLNWSRTEOTOAL
AHIEAWITNNWRIS
and write the letters, alternating from each row.

The decoded article is

<u>Take a chance on me</u>
If you change your mind / I'm the first in line / Honey I'm still free / Take a chance on me / If you need me / Let me know / Gonna be around / If you got no place to go / If you're feeling down / If you're all alone / When the pretty birds are flown / Gonna do my very best and it ain't no lie / If you put me to the test / If you let me try

CIRCLE PARTS AND REGIONS

Diameters

Number of diameters (d)	Number of regions (R)
1	3
2	5
3	7
4	9
5	11
6	13

$R = 2d + 1$

Chords

Number of chords (c)	Number of regions (R)
1	3
2	5
3	8
4	12
5	17
6	23

$R = \frac{n(n+1)}{2} + 2$

Tangents

Number of tangents (t)	Number of regions (R)
1	3
2	6
3	10
4	15
5	21
6	28

$R = \frac{(t+1)(t+2)}{2}$

Secants

Number of secants (s)	Number of regions (R)
1	4
2	8
3	13
4	19
5	26
6	34

$R = \frac{(s+2)(s+3)}{2} - 2$

MISSING PIECE PUZZLES

Don't assume that the diagrams are all the same side length.

If a small square in the first diagram has length x units, the side of the top square is $2\sqrt{2}x = 2.828 \ldots x$ and that is less than the side of the second diagram which is $3x$.

Similarly, the second diagram of the second pair has a larger side.

PUZZLE PAGE

Easy

5	4	1	4
1	2	5	3
4	3	1	2
1	5	4	3
3	2	1	2

Medium

3	1	2	1
2	4	3	4
5	1	2	1
2	4	5	4
1	3	2	3

Harder

2	4	2	1	2	1	3	1	4
1	3	5	3	4	5	2	5	2
5	2	1	2	1	3	4	3	1
1	4	5	4	5	2	1	5	2
3	2	3	1	3	4	3	4	1

Very hard

1	5	2	1	3	5	4	3	4
2	3	4	5	2	1	2	5	2
4	1	2	1	4	5	3	1	3
2	5	4	5	2	1	2	5	4
3	1	3	1	3	4	3	1	2

SIMPLY SOLVE

Q1 231	Q2 31	Q3 11	Q4 496, the third perfect number
Q5 30	Q6 28	Q7 47	Q8 20
Q9 3135	Q10 12	Q11 3	Q12 2
Q13 48	Q14 1997	Q15 £5.25	Q16 26 birds, 17 beasts

STUDENT PROBLEMS

Problem 1992.1

x, y and z are positive real numbers with $xy + yz + zx = 1$. Show that

$$x + y + z \geqslant \sqrt{3}.$$

Solution

(a) The following approach was generally taken:

$$(x+y+z)^2 - 3 = (x+y+z)^2 - 3(xy+yz+zx) \quad \text{(since } xy + yz + zx = 1\text{)}$$
$$= x^2 + y^2 + z^2 - (xy + yz + zx)$$
$$= \tfrac{1}{2}\left[(x-y)^2 + (y-z)^2 + (z-x)^2\right] \geqslant 0.$$

The required result now follows easily.

Problem 1992.2

What are the last two (right-hand) digits of the number 19^{92} when written in the usual decimal notation?

Solution

The response was a little disappointing to problems 1992.2 and 1992.3. Brian Fulthorpe (Prudhoe County High School), Justin Woo (Stonyhurst College), Matthew James (Repton School) and Jan Gitowski (Stonyhurst College) correctly found the final two digits of 19^{92} which are 61; Jan by noting the final terms of the expansion of $(20 - 1)^{92}$, while the others worked modulo 100 along the following lines:

$$19^1 \equiv 19,\ 19^2 \equiv 61,\ 19^3 \equiv 59,\ 19^4 \equiv 21 \text{ and } 19^5 \equiv -1 \pmod{100}.$$

Thus $19^{10} = (19^5)^2 \equiv (-1)^2 = 1$ and $19^{92} = (19^{10})^9.19^2 \equiv 1^9.61 = 61$, so that 19^{92} ends ... 61.

Problem 1992.3

Give, with working, the prime factorisation of the twenty-digit number 12 318 876 808 768 112 319. You may quote in your solution the results of using a non-programmable pocket calculator.

For 1992.3, Brian Fulthorpe appeared to be the only student to have reached the correct answer, although (sadly) it seems only as a result of commendable perseverance on a pocket calculator. It should be expected that all GCSE and A level students can test (with the aid of a pocket calculator) divisibility by primes up to 100, but (realistically) not much further. If the given twenty-digit number is N then, testing for divisibility by such low primes gives $N = 3^4.11.41^2.97.M$, where M is the 11-digit number 84 791 820 637 – still large enough to exceed most calculators' displays. Since further direct attack is unrealistic, something more subtle might be appropriate. The £20 award was delayed in the hope of receiving further efforts for the second problem and the following hint was given:

N = 12 318 876 808 768 112 319. Let n = 12319. Then N = ...

ANSWERS FROM ISSUE 69

CROSSNUMBER

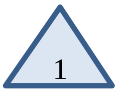
5	4	3	2	■	8
0	■	■	4	2	1
5	■	9	9	7	■
■	7	3	■	9	7
4	2	■	■	■	8
1	■	1	7	2	9

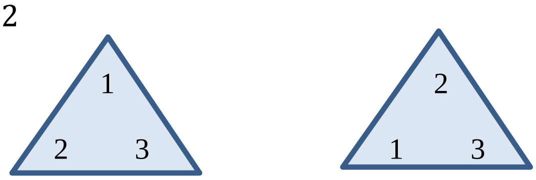
IN SEARCH OF A-P TRIANGLES

Each solution can be reflected in a vertical axis.

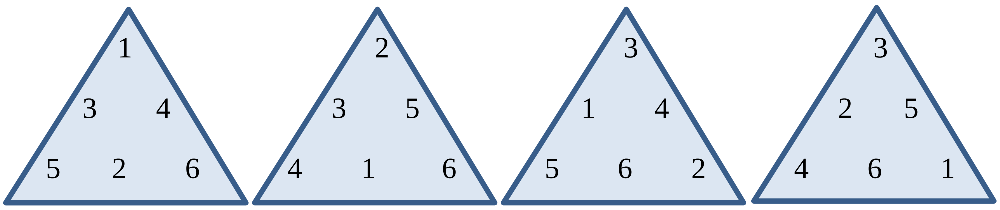
$n = 2$

$n = 1$

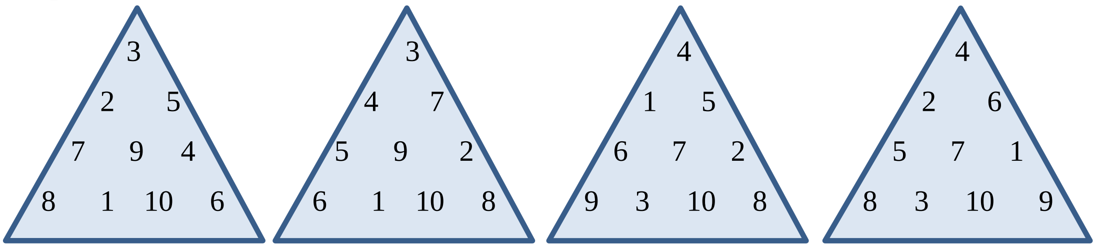




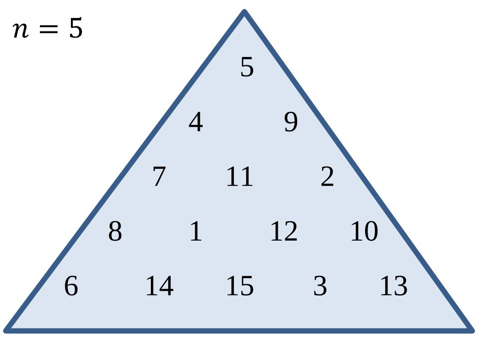
$n = 3$



$n = 4$



$n = 5$



THAT CERTAIN SUM THING

Erick thinks that answers are not needed for this as if you have done it you know whether you have solved it. However, I have included the answers to show that there are no mistakes in the puzzles and they can be solved!

3	5	2	3	13
1	4	1	1	7
4	5	3	4	16
2	5	1	2	10
4	5	2	3	14
14	24	9	13	

4	3	4	1	3	15
1	2	4	1	2	10
5	5	5	2	5	22
3	2	4	1	3	13
3	2	5	1	4	15
16	14	22	6	17	

3	1	4	2	10
5	2	5	5	17
3	1	4	3	11
4	1	5	3	13
2	1	4	2	9
17	6	22	15	

5	4	1	5	3	18
2	3	2	4	2	13
1	1	1	2	1	6
5	4	3	5	4	21
4	3	2	5	3	17
17	15	9	21	13	

3	1	5	3	12
4	2	5	4	15
2	1	2	2	7
3	1	5	3	12
4	1	5	4	14
16	6	22	16	

2	1	1	3	4	1	12
5	4	2	5	5	5	26
3	3	1	4	4	3	18
3	2	1	3	4	2	15
4	2	1	5	5	2	19
17	12	6	20	22	13	

5	3	4	4	5	4	25
3	2	3	2	5	3	18
1	1	1	1	2	1	7
2	2	3	1	4	2	14
5	4	5	4	5	3	26
16	12	16	12	21	13	

KNIGHT'S TOURS - 2

Andrew Palfreyman has sent in the following observation about generalising the result for a 3 by *n* grid.

In my article in *SYMmetry*plus issue no. 68 it was noted that the smallest length grid with three rows for which a knight's tour exists is the 3 by 4. In that issue, the 3 by 5 grid was shown not to have a knight's tour and the 3 by 6 grid was shown not to have one either in issue no. 69.

Here are possible knight's tours for the grids from 3 by 7 to 3 by 10:

17	14	19	2	5	8	11
20	1	16	13	10	3	6
15	18	21	4	7	12	9

5	2	19	22	9	12	17	14
20	23	6	3	18	15	8	11
1	4	21	24	7	10	13	16

5	2	17	14	11	8	19	22	25
16	13	6	3	18	27	24	9	20
1	4	15	12	7	10	21	26	23

5	2	27	24	13	8	11	20	15	18
30	25	6	3	28	23	14	17	10	21
1	4	29	26	7	12	9	22	19	16

Now consider this particular knight's tour for a 3 by 4 grid:

3	6	9	12
8	11	2	5
1	4	7	10

Imagine that this is placed to the left of the 3 by 7 grid above. Furthermore, let us increase all the values in the 3 by 7 grid by 12. We thus have the following arrangement:

3	6	9	12	29	26	31	14	17	20	23
8	11	2	5	32	13	28	25	22	15	18
1	4	7	10	27	30	33	16	19	24	21

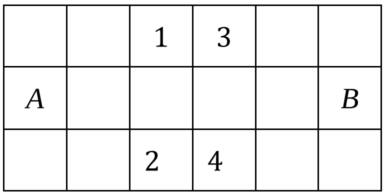
We have thus created a 3 by 11 knight's tour out of the 3 by 7 one!

On inspection, it can be observed that the 3 by 8, 3 by 9 and 3 by 10 knight's tours will also generate solutions for the 3 by 12, 3 by 13 and 3 by 14 grids by the same method.

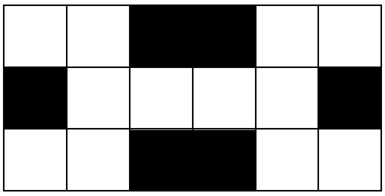
Moreover, given that the knight starts in the bottom left hand corner in each of these new grids (i.e. the 3 by 11, 3 by 12, 3 by 13 and 3 by 14), the process can be repeated by placing another 3 by 4 to the left of each of them and adding 12 to all their positions to gain knight's tours for grids of dimensions 3 by 15, 3 by 16, 3 by 17 and 3 by 18.

I will not give a formal proof by mathematical induction here, but I hope that it can be seen that this process can be repeated *ad infinitum* and so a grid with three rows will have a knight's tour so long as the number of columns is either 4, or greater than 6.

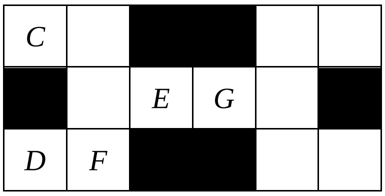
Wil Ransome sent in the following: I think I have improved on the 3 by 6 case slightly.



A must go to 1 and 2; B must go to 3 and 4, which leads to:



Now, in the diagram below, C and D cannot both go to E, so assume C goes to F. This forces G. Now there are never three more successive possible moves from G.



PERFECT TRIANGLES

Graham Hoare who let me know of two errors I made when processing his article. In the first line, rather than 'In this brief note we connect **to** items that appeared …' I should have written 'In this brief note we connect **two** items that appeared …'.

The other error is a far more serious one. In the third paragraph, I wrote 'Let us remind ourselves of **Euclid**'s result, namely, that an <u>even</u> perfect number must have the form $2^{p-1}(2^p-1)$, where 2^p-1 is prime.' This is not Euclid's result, but **Euler**'s result.

My sincere apologies to all three gentlemen (but most of all Graham) for these mistakes.

SOME FRIEZES

Bob Burn's article first appeared on the Association of Teachers of Mathematics website, but is no longer there. He has confirmed that it is his and part of a larger package on Friezes that he has permission to do with it as he pleases. If readers are interested in reading more, then please let me know at symmetryplus@m-a.org.uk

SIMPLY SOLVE - 2

Q1 80	Q2 4000	Q3 2p, 18p	Q4 190
Q5 16.5	Q6 60, 30 and 10kg	Q7 2	Q8 2.8m
Q9 110	Q10 $21 \div 7 + 8 = 11$	Q11 The letter M	Q12 11:59
Q13 91	Q14 1240	Q15 8.75	Q16 13 and 15

Q17 3157 – the numbers are 3125 and 32

ANSWERS FROM ISSUE 70

PERIODIC SEQUENCES

The possibilities are as follows.

- If either initial value is 0 then the process will end with an attempted division by zero.
- If both initial values are positive then the sequence will be cyclic of period 6 unless both initial values are 1, when the cycle is (1).
- If the initial values are both negative or one negative and one positive then the sequence will be cyclic of period 21 unless both initial values are ± 1 when the sequence will be $(-1, 1, -1)$.

For an even greater challenge, investigate the behaviour of sequences defined by $xz = \max(y, \beta)$ for $\beta \neq -1$.

THE FOUR FIVES CHALLENGE

Here are some solutions. There will, of course, be more.

$0 = 5 + 5 - 5 - 5$ or $(5+5)\times(5-5)$

$1 = (5+5-5)\div 5$ or $(5\div 5)\times(5\div 5)$

$2 = 5\div 5 + 5\div 5$ or $(\sqrt{5\times 5}+5)\div 5$

$3 = (5+5+5)\div 5$

$4 = (5\times 5 - 5)\div 5$

$5 = \sqrt{5\times 5} + 5 - 5$

$6 = (5\times 5+5)\div 5$ or $55\div 5 - 5$

$7 = 5 + (5+5)\div 5$

$8 = \dfrac{5!}{5+5+5}$

$9 = 5\div .5 - 5\div 5$

$10 = (55-5)\div 5$

$11 = 5+5+5\div 5$

$12 = (55+5)\div 5$

$15 = 5\times 5 - 5 - 5$

$20 = 5+5+5+5$

$25 = 5\times 5 + 5 - 5$

$30 = 5\times 5 + \sqrt{5\times 5}$

$35 = 5\times 5 + 5 + 5$

$40 = \dfrac{5\times 5 - 5}{.5}$

$45 = 5\times 5\div .5 - 5$

$50 = 5\times 5 + 5\times 5$

$55 = 55 + 5 - 5$

$60 = 55 + \sqrt{5\times 5}$

$65 = 55 + 5 + 5$

STUDENT PROBLEMS

Problem 1993.1

Show that the equation $x^3 + y^3 = z^3 + 1993$ has no solutions in integers x, y, z.

Solution

To show that an equation has no solutions in integers, a very effective way to proceed is to work in arithmetic modulo n, for some well chosen value or values of n. Successful solvers all chose to work modulo 9 along the following lines:

For integers n, $n^3 \equiv -1, 0, 1 \pmod 9$ so that $x^3 + y^3 \equiv 0, \pm 1, \pm 2 \pmod 9$, while $z^3 + 1993 \equiv \{0, \pm 1\} + 4 \equiv 3, 4, 5 \pmod 9$. So the LHS and RHS of the equation never leave the same remainder upon division by 9 for integers x, y, z, and there are no solutions in integers.

Problem 1993.2

A straight line is drawn through one corner, A, of a parallelogram, to meet the diagonal and the two sides which do not pass through A (produced, if necessary) at P, Q and R respectively. Prove that $AP^2 = PQ.PR$

Solution

Firstly it is worth noting that there are a number of different cases:

(i) If the line through A is parallel to the diagonal not through A, then there is no intersection point P. The wording of the problem precludes this possibility.

(ii) If the line through A passes through C (the opposite corner of the parallelogram) then $Q = R = C$ and, since the diagonals of a parallelogram bisect each other, $AP = PQ = PR$ and the result is trivially true.

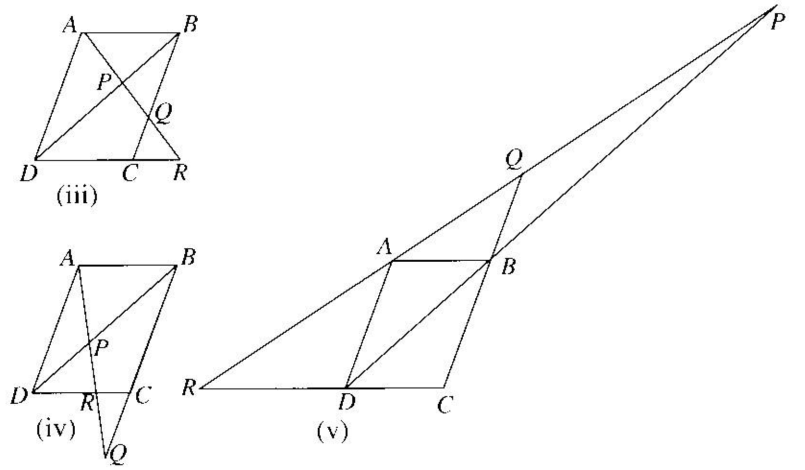
(iii) The line passes between B and C. $\angle PAB = \angle PRD$ (alternate angles $AB \parallel DR$); $\angle APB = \angle RPD$ (vertically opposite angles). So triangles APB, RPD are similar and $AP/RP = PB/PD$ (I).

$\angle QBP = \angle ADP$ (alternate angles $BQ \parallel AD$); $\angle APD = \angle QPB$ (vertically opposite angles). So triangles APD, QPB are similar and $QP/AP = PB/PD$ (II).

(I) and (II) together give $AP/PR = PQ/AP$ and $AP^2 = PQ.PR$.

(iv) The line passes between C and D. This is equivalent to case (iii), with the roles of Q and R reversed, which does not affect the result.

(v) The line lies outside the parallelogram. Again triangles APB, RPD are similar, as, for different reasons are triangles APD, QPB.



One case nobody mentioned was if the line through A should pass through either of the sides AB or AD, in which case $P = Q = R$ (= B or D) and the result is again trivially true.

An alternative approach to this problem is to use vectors, which has the merit of covering all the cases at once, but it is clumsy, and the details are left to the interested (and masochistic) reader.

Problem 1993.3

Find all the values of n for which the sum $25+33+41+\ldots+(8n+17)$ is a perfect square.

Solutions:

Method I (David Hotham)

The series is the sum of an A.P. for which the formula gives $n(4n+21)$. Let $4n^2+21n=k^2$ for some positive integer k. Multiplying through by 16 and completing the square, we have

$$(8n+21)^2-16k^2=441 \Rightarrow (8n+21-4k)(8n+21+4k)=441,$$

by the factorisation of a difference of two squares. Both factors are integers, the first clearly less than the second. Suitable factorisations of 441 are 1×441, 3×147, 7×63 and 9×49. Solving the resulting pairs of simultaneous equations yields, respectively, $n=25$, $6\frac{3}{4}$, $1\frac{3}{4}$ and 1. There are thus the two possible positive integer values of n.

Method II (Richard Davies)

Again $4n^2+21n=k^2$ for some positive integer k. Since $n>0$, $k^2>4n^2$ and $k>2n$. Set $k=2n+a$ for some positive integer a. Then

$$4n^2+21n=(2n+a)^2 \Rightarrow a^2=(21-4a)n.$$

As the RHS of this equation is positive, $21-4a>0$ and $1\leqslant a\leqslant 5$. Trying these few possibilities again yields $n=1$ and $n=25$.

Many of the other solvers produced similar reasoning somewhere along the line, although a number of lengthier arguments devoted themselves to studying the possible highest common factors of n and $(4n+21)$. It is worth noting that the problem of solving

$$(8n+21)^2=21^2+(4k)^2$$

is equivalent to finding integer sided right-angled triangles having a short side of length 21 with one or two extra restrictions. The case $n=1$ corresponds to the (20, 21, 29) triangle, while $n=25$ corresponds to the (21, 220, 221) triangle.

Problem 1993.4

For real numbers x, y, z, prove that $\left(\frac{x}{2}+\frac{y}{3}+\frac{z}{6}\right)^2\le\frac{x^2}{2}+\frac{y^2}{3}+\frac{z^2}{6}$.

When does equality occur?

Solution:

The sheer variety of approaches used was marvellous to witness. Just one is shown here.

Method I

Completing the square and noting that squares are non-negative yields

(a) (David Hotham, Paul Heslop)

$$\frac{x^2}{2}+\frac{y^2}{3}+\frac{z^2}{6}-\left(\frac{x}{2}+\frac{y}{3}+\frac{z}{6}\right)^2$$
$$=\tfrac{1}{6}(x-y)^2+\tfrac{1}{12}(x-z)^2+\tfrac{1}{18}(y-z)^2\geqslant 0$$

with equality when each bracketed term is zero, i.e. $x=y=z$.

Or

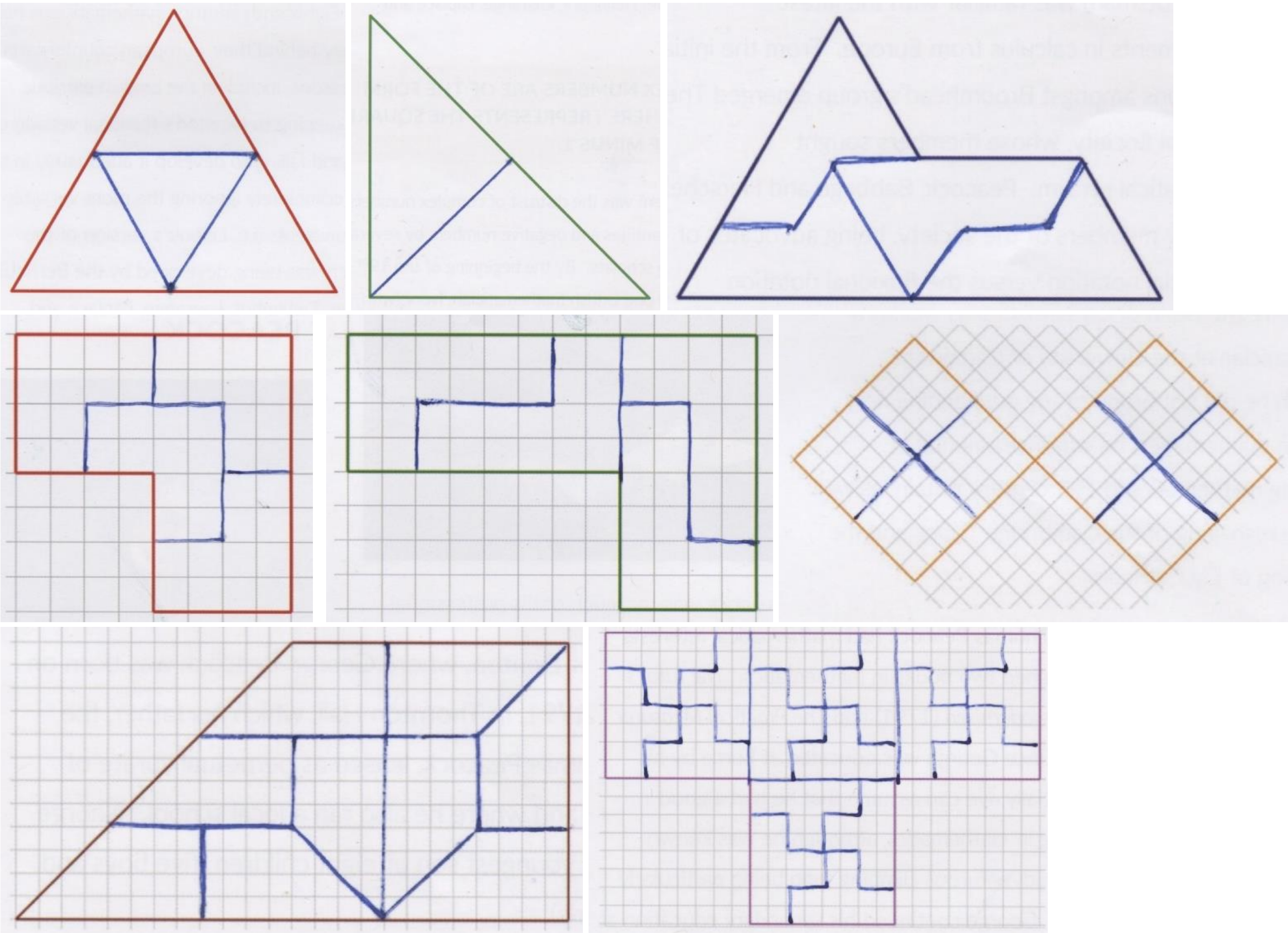
(b) (Edmund Jones, Tim Riley, Jeremy Collins, Michelle Hawke)

$$\frac{x^2}{2}+\frac{y^2}{3}+\frac{z^2}{6}-\left(\frac{x}{2}+\frac{y}{3}+\frac{z}{6}\right)^2$$

$$= \tfrac{1}{36}(3x - 2y - z)^2 + \tfrac{1}{9}(y - z)^2 \geqslant 0$$

again with equality when $x = y = z$.

REPTILES



QUICKIE 37

Dumbo, Bambi, Doc, Dopey and Pluto, in that order

THE GENIUS OF GREAT MATHEMATICIANS

Mathematician who solved Fermat's Last Theorem	Andrew Wiles
Mathematician who invented Cartesian co-ordinates	Rene Descartes
Mathematician who invented the zeta function	Georg Bernhard Riemann
Mathematician who invented the topic we now know as topology	Leonhard Euler
Had no formal education in maths but made substantial contribution to many areas	Srinivasa Ramanujuan
Mathematician who was able to show that some theorems cannot be proved	Kurt Gödel
A mathematician who hinted at the existence of imaginary numbers in his book *Ars Magna*	Gerolamo Cardano
Wrote *Ilm al-jabr wa'l muqabalah*, which introduced what came to be called algebra	Al'Khwarizmi
A prolific Hungarian mathematician who published over 1500 papers, notably on number theory	Paul Erdos
Mathematician who the first person to prove the fundamental theorem of arithmetic	Carl Frederick Gauss
A highly influential mathematician in the development of theoretical computer science	Alan Turing (1912-54)
The first woman known to have been a mathematician	Hypatia
Mathematician who proved that every positive integer is the sum of four squares	Joseph-Louis Lagrange

SIMPLY SOLVE - 3

Q1 105 = 3 x 5 x 6 Q2 10 and 15

Q3 9 as it can appear in four ways (6,3; 5,4; 4,5; 3,6). 10 only appears in three ways (6,4; 5,5; 4,6)

Q4 Sarah is 33, her son is 11. To solve this, let her son's age now by x years. Sarah is then $3x$ year's old. In 12 year's time Sarah is $3x + 12$ and her son is $x + 12$. The resulting equation is $3x + 12 = 2(x + 12) - 1$ and solving that equation gives the answer.

Q5 P(2H) = 0.25, P(Picture card if Ace is included) = 4/13 > 0.25, P(Picture card if Ace is not included) = 3/13 < 0.25

Q6 73037 Q7 $6 \div (1 - 5 \div 7) = 21$

Q8 Place two chairs on each wall and put one of each remaining chairs at each end of one of the diagonals

Q9 9, 81, 576, 2304 Q10 29 chickens, 19 dogs

Q11 e.g. 7, 8, 11, 12, 12 Q12 At the corners of a regular tetrahedron

Q13 93 and 87: the multiples of 13 with their digits reversed

Q14 8 pens Q15 45 cm Q16 36 feet Q17 80

ANSWERS FROM ISSUE 71

CROSSNUMBER

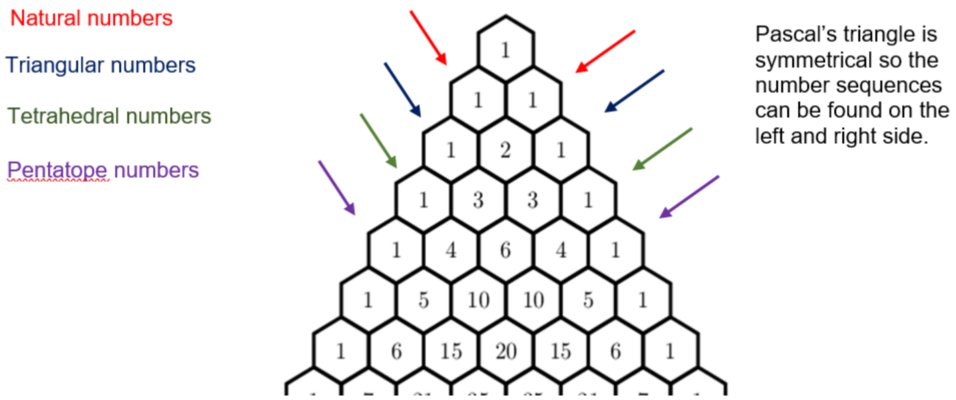
8	1		1	4	4
0		1	2	5	
6	1	6		6	
	0		5	7	6
	2	2	4		1
1	4	3		4	2

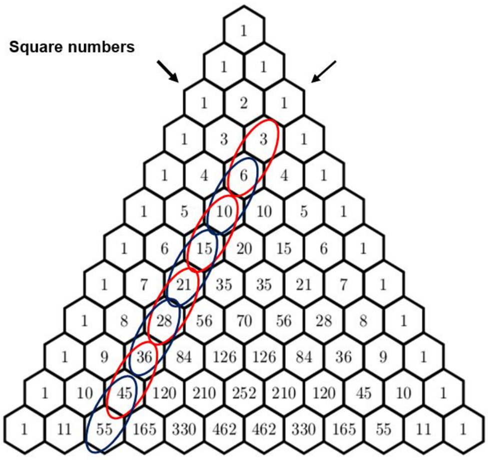
EGOTISTICAL NUMBERS

Eagle-eyed readers will realise that Erick missed 2020 as an egotistical number at the bottom of the first column. It is the only egotistical year we will live in! Here is a list of the egotistical ones: 1210, 2020, 21200, 3211000, 42101000, 521001000, 6210001000.

NUMBER ARRAYS AND NUMBER PATTERNS

Array 1





Sum the numbers in each loop.

This shows that square numbers are produced from the sum of two consecutive triangle numbers.

Proof: $\frac{n(n-1)}{2}+\frac{(n+1)n}{2}=\frac{n^2-n+n^2+n}{2}=\frac{2n^2}{n}=n^2$.

It is also possible to find the square numbers by taking the differences of numbers two places apart on the 4th diagonal of Pascal's Triangle.

4th diagonal (tetrahedral numbers): 1, 4, 10, 20, 35, 56, 84, 120, 165 …

$1-0=1=1^2$

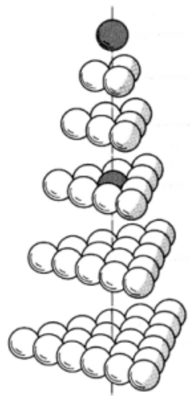
$4-0=4=2^2$

$10-1=9=3^2$

$20-4=16=4^2$

…

Tetrahedral numbers



Tetrahedral numbers are derived from the sum of the first *n* triangular numbers.

$1=1$

$1+3=4$

$1+3+6=10$

$1+3+6+10=20$

…

Cube numbers

Finding the cubes number sequence requires using the tetrahedral numbers.

1. List the tetrahedral numbers: 1, 4, 10, 20, 35, 56, 84, …
2. Place two zeros at the start of the sequence, giving: 0, 0, 1, 4, 10, 20, 35, 56, 84, …
3. Now, add three consecutive numbers at a time, multiplying the middle number by 4.

Applying the rule to the first seven cubes numbers

$0+4(0)+1=1=1^3$

$0+4(1)+4=8=2^3$

$1+4(4)+10=27=3^3$

$4+4(10)+20=64=4^3$

$10+4(20)+35=125=5^3$

…

Ref: Alec Shute, *Pascals Triangle and Cube Numbers* (Education Articles, September 2014)
(There are several accessible articles by this author on sequences to be found in Pascal's Triangle.)

Rule for each sequence.

Square numbers; n^2

Cube number: n^3

Triangular numbers: $\frac{n(n+1)}{2}$; Tetrahedral numbers: $\frac{n(n+1)(n+2)}{3!}$; Pentatope numbers: $\frac{n(n+1)(n+2)(n+3)}{4!}$; Sequence 1, 6, 21, 56, 126, … $\frac{n(n+1)(n+2)(n+3)(n+4)}{5!}$.

The growth of each rule is a good example of the beauty of mathematics.

Further tasks

1. Find the Fibonacci sequence within the triangle.
2. Explore the patterns produced by multiples.

Array 2

The sum of each line gives the cube numbers.

The fifteenth row will be $15^3 = 3375$.

Array 3

The bottom number in each column is a square number (n^2).

The product of any two adjacent numbers in a row is itself in the row, e.g. $5 \times 11 = 55$. The product is n places to the right of the smaller number, e.g. 55 is five places to the right of 5.

Second example, $4 \times 8 = 32$, which is four places to the right of the 4.

In the longest row (centre line), the numbers are of the form $n^2 - n + 1$, where n is 1, 2, 3, 4, 5, …. For example:

$1^2-1+1=1$

$2^2-2+1=3$

$3^2-3+1=7$

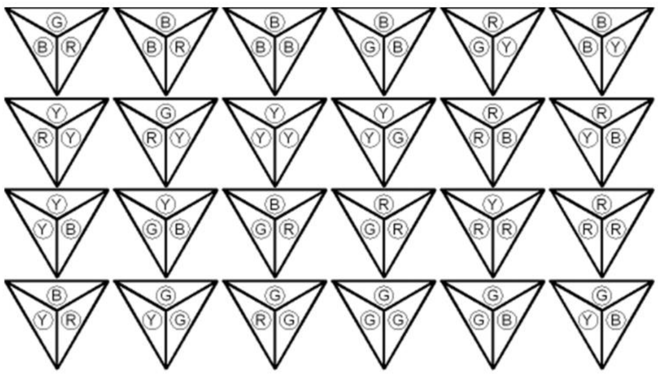
MACMAHON'S COLOUR TRIANGLES

1 With three colours there are 11 distinct triangles.

2 With four colours there are 24 distinct triangles.

An interesting activity with all 24 triangles is to arrange them to form a regular hexagon with touching sides the same colour.

With five colours there are 45 distinct triangles.



3 $\frac{n(n^2+2)}{3}$

4 With ten colours there are 340 distinct triangles.

TRANSLATING TRIANGLES

For all the answers below, the $\begin{pmatrix}0\\0\end{pmatrix}$ translation has been counted. If you do not count that as a proper translation, then the number of translations in answers 1 to 4 below should be reduced by 1 and the grid in Q5 could only be 201×2 or 3×200.

The $m \times n$ grid is defined as having m rows and n columns.

1 For a 3×3 grid there are, there are two translations possible: $\begin{pmatrix}0\\0\end{pmatrix}$ and $\begin{pmatrix}1\\0\end{pmatrix}$.

2 For a 4×4 grid there are 6 possible translations; for a 5×6 grid, 15 translations; for a 6×5 grid, 16 translations.

3 $(m-2)(n-1)$

4 110, 150

5 The grid could be any of the following sizes: $200 \times 2, 101 \times 3, 68 \times 4, 35 \times 7, 24 \times 10, 20 \times 12, 13 \times 19,$ $11 \times 23, 8 \times 34, 5 \times 67, 4 \times 100, 3 \times 199.$

357 ANCIENT CHINESE PUZZLE

Move as follows: 1. Rd6 Kc8 2. Ka7 Kc7 3. Rc6 mate

77 DIGITS AND SQUARES

2	1	9
4	3	8
6	5	7

2	7	3
5	4	6
8	1	9

3	2	7
6	5	4
9	8	1

SIMPLY SOLVE - 4

Q1 6 ways: 97 + 3, 89 + 11, 83 + 17, 71 + 29, 59 + 41, 53 + 47

Q2 Letters above the line are composed of straight-line segments, those below have curved parts

Q3 210 – the 20th triangle number

Q4 10, ${}^5C_2$ combinations possible

Q5 Let x be the age of the middle child in years, then $(x+3)+x+(x-3)=57$, so $x=19$ and ages are 18, 19 and 22

Q6 15.6 cm

Q7 2 hours. In one hour, Sharon completes $\frac{1}{3}$ of the job and Carl $\frac{1}{6}$ of the job, so working together for an hour they complete $\frac{1}{2}$ the job.

Q8 102.3 cm

Q9 4, 2, 7 Let the owls catch a, b, c worms respectively. Then $a+b=6, b+c=9, a+c=11$. Adding all three equations gives $2a+2b+2c=26$, so $a+b+c=13$. Substitute the original three equations into this one to obtain the answers.

Q10 10 This is because $x+y=14, a+b+c=36$ so the five values add to 50.

Q11 2520

Q12 Jade: £180, Kellie £100, Len: £50 Since J:K:L = 18:10:5 and each part will be $£330 \div (18+10+5) = £10$

Q13 4 hours

Q14 35 and 62

Q15 144cm^2 because the rectangle is 24cm by 6cm

Q16 8

ANSWERS FROM ISSUE 72

SPIRAL LENGTH

The length of the spiral will form the hypotenuse of a right-angled triangle with sides of length 225mm and $46\pi \times 2.5$ mm. using Pythagoras' Theorem this means the spiral is 430mm long to 2 sig.fig..

FIGURE IT OUT

3	3	2	1	0		6	1	7	4
7		2	5		3	7	9		8
1	5		2	1	6		0		4
	5	9	1	3		7		3	
8		1		5	6		9	6	1
1	4		2		1	3	4	9	
2	4		8	5		1			9
8		3	4		7	5	6	0	

ACROSS
1 135×246 135 The digits in the hundred's column need to be the smallest, while those in the unit's column, the largest.
7 25 = 33 – 2
11 33 + 43 + 53 = 27 + 64 = 125 = 216 = 63. An example of the beauty of mathematics.
15 $6s^2$ is the formula. Let s be a side. $6s^2 = 294$, so $s^2 = 49$ and $s = 7$
20 The digits reversed are 169, which is 13^2. There are, however, other square numbers which are squares when reversed: 121 and 144, but the other clues prevent alternate answers.
26 If the distance between the two places is d, then the total time taken to do the journey there and back is $\frac{d}{30} + \frac{d}{20}$ and the average speed is total distance over total time, i.e. $\frac{2d}{\frac{d}{30} + \frac{d}{20}}$, which makes the average speed 24mph. This is an example of the harmonic mean. The harmonic mean is the reciprocal of the arithmetic mean of the reciprocals.
27 85 = 92 + 22 = 72 + 62
28 $\frac{2x}{3} = 6 + \frac{x}{2}$, so $\frac{x}{6} = 6$ and x = 36. One quarter of this is 9.
29 The 4 by 4 magic square uses the numbers from 1 to 16 inclusive and has a Magic Constant of 34. The formula for finding the constant or magic sum is $\frac{n(n^2+1)}{2}$ where n represents the length of the square.

Down
1 $371 = 3^3 + 7^3 + 1^3$
Other three-digit numbers that equal the sum of the cubes of their own digits are 153, 370 and 407. An n-digit number that is the sum of the n^{th} powers of its digits is called a narcissistic number.
12 $1^1 + 3^2 + 5^3 = 135$. Note the rising powers. There are other such examples: $1^1 + 7^2 + 5^3 = 175$ and $5^1 + 9^2 + 8^3 = 598$.
23 Amicable numbers are two different numbers so related that the sum of the proper divisors of each is equal to

the other number. The smallest pair of amicable numbers is (220, 284). They are amicable because the proper divisors of 220 are 1, 2, 4, 5, 10, 11, 20, 22, 44, 55 and 110, of which the sum is 284; and the proper divisors of 284 are 1, 2, 4, 71 and 142, of which the sum is 220. The next pair are 1184 and 1210.

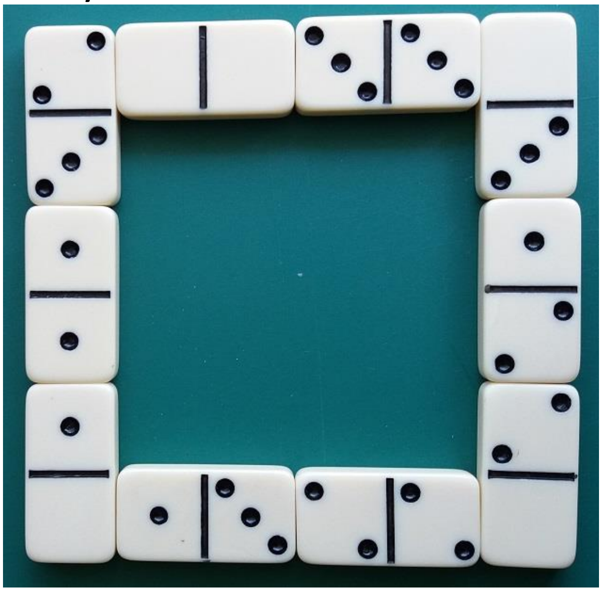
COUNTING AND THE THEORY OF NUMBERS

To be able to talk about a number being odd or even it is necessary for us to show that the number is finite. Rewriting the equation as $p = x^2 + 2x(y+z) + (y-z)^2$ makes this obvious since x, y and z must all be less than $\frac{p}{2}$.

MATHEMATICS WITH DOMINOES

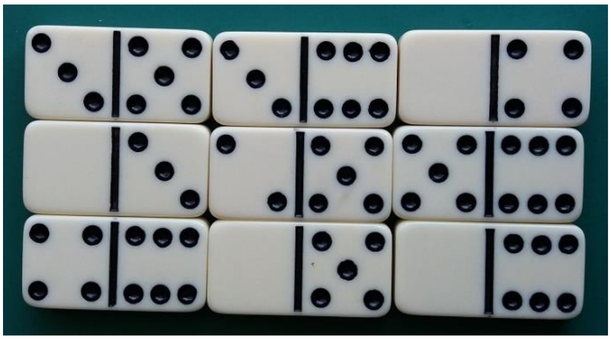
Activity 1
90
Multiply the numbers on the left and multiply those on the right. Now add them.
$(3\times5\times2)+(6\times5\times2)=30+60=90$

Activity 2



Activity 3

Activity 4



$12\times3\div9=4$, so central domino has to have 4 dots.

$21\times3\div9=7$, so central domino has to have 7 dots.

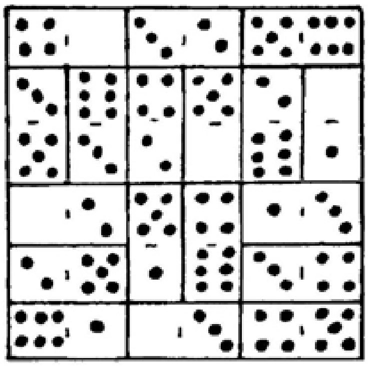
Activity 5
The puzzle is impossible to complete. A domino placed on the chessboard will always cover one white square and one black square. Therefore, a collection of dominoes placed on the board will cover an equal number of squares of each colour. If the two white corners are removed from the board then 30 white squares and 32 black squares remain to be covered by dominoes, so this is impossible. If the two black corners are removed instead, then 32 white squares and 30 black squares remain, so it is again impossible.

Activity 6

In a double n set of dominoes, there are $\frac{(n+1)(n+2)}{2}$ dominoes and the total number of dots/pips is $\frac{n(n+1)(n+2)}{2}$.

So, the answers are a) 66 b) 91 c) 2040.

Activity 7



This is one solution that uses all 18 dominoes.

There are, of course, other solutions.

3 X 3 GRIDS

Activity 1

This is an example of a Latin Square.

1	2	3
3	1	2
2	3	1

Activity 2

In total there are 22.

Six 2 by 1s, and six 1 by 2s. Three 3 by 1s, and three 1 by 3s. Two 3 by 2s, and two 2 by 3s.

So, 12 + 6 + 4 = 22

Activity 3

16 moves are required. The earliest record of this problem, known as Guarini's problem, was found in 1512 in Europe. The moves can be thought of as four groups of moves in which the four knights move from corner square to middle square, and vice-versa in rotation.

This problem can be solved using Graph theory.

Ref: https://www.youtube.com/watch?v=QlZKvDxCHvg

Activity 4

Factorise the products to find answers.

e.g. $21 = 1 \times 3 \times 7$, $24 = 2^3 \times 3$ So, the 3 must go in the middle square of the top row.

2	3	4
5	1	8
6	7	9

Activity 5
First find the magic constant, which is $59 \times 3 = 177$.
Thereafter, basic addition and subtraction will allow the completion of the magic square.

47	29	101
113	59	5
17	89	71

Activity 6
Using just one colour, which means whole 1 by 1 squares.
Each of the nine 1 by 1 squares can be either coloured or not.
That means there are two choices for the first square – coloured or not coloured.
Then there are two choices for the next square, and so on. That gives $2^9 = 512$ possible patterns in total.

If two colours are used (e.g. blue and green), then each of the nine 1 by 1 squares can be coloured blue, green, or left uncoloured. There are three choices for each square. That gives $3^9 = 19\ 683$ possible patterns in total.

CREATING SYMMETRY … PLUS?

$c^2 = a^2 + b^2 - 2ab$

SIMPLY SOLVE - 5

Q1 25. Let her sister's age now be x. Mel's age now is therefore $3x$. Five years ago, her sister would be $x - 5$ years old and Mel would be $3x - 5$. Since the sum of their ages then was 18, we have $x - 5 + 3x - 5 = 18$. Solving this gives $x = 7$ and so Mel is now 21 and will be 25 in four years' time.

Q2 $280 = 2^3 \times 5 \times 7$, and since their ages add to 20, they must be 5, 7 and 8 years old. In 12 years' time they will be 17, 19 and 20. $17 \times 19 \times 20 = 6460$.

Q3 240km. Distance = speed x time, so distance $= 180 \times 1\frac{1}{3} = 240$ Q4 $87 = 2^2 + 3^2 + 5^2 + 7^2$

Q5 This problem can be solved in several ways, such as trial and improvement and using a table. A better mathematical solution is to use algebra. From the information given in the question, it is possible to produce two equations. Let the number of pigs be p, the number of goats be g and the number of sheep be s.
So, $p + g + s = 100$ (1) and $21p + 8g + 3s = 600$ (2)
The difficulty now is that there are two equations and three unknowns. Ideally, three equations are required. We do know however, that the solutions are integers and having a fraction of an animal is not possible. (These are examples of Diophantine equations).
We can eliminate s from the two equations by multiplying (1) by 3 to obtain equation (3) and subtracting:
$3p + 3g + 3s = 300$ (3)
$21p + 8g + 3s = 600$ (2)
$18p + 5g = 300$
So, $g = \frac{300-18p}{5}$.(4)
We know that there must be one of each animal, that p is even and 5 must be a factor of $18p$, so p is a multiple of 10. Putting $p = 10$ into equation (4) gives 24 goats, 10 pigs and
$100 - 24 - 10 = 66$ sheep.
When 20 is substituted into the equation (the next multiple of 10), the answer is negative, which is not possible, so the solution is 10 pigs, 24 goats, 66 sheep.

Q6 3 Adding the two equations gives $A = 57$, so $B = 19$ and result follows Q7 One way is $99 + (99 \div 99)$.

Q8 $P(Y) \times P(R) = \frac{5}{12} \times \frac{7}{11} = \frac{35}{132}$ Q9 $180(n-2) = 2340$, so $n = 15$. 15 sides

Q10 Let the number be x. Then $\frac{x}{x} = 2x$, i.e. $1 = 2x$, so $x = \frac{1}{2}$. Q11 $11^3 = 1331$ The clue is in the question number!

Q12 78, 325. The formula for the triangle numbers is $\frac{n(n+1)}{2}$, where n is the number of rows. Q13 21

Q14 10:54:19 Q15 40 children. Q16 The unit digits of the powers of 7 cycle through $7, 9, 3, 1$. 2020 has a remainder of 0 when divided by 4, so the unit digit of 7^{2020} is the same as 7^0, i.e. 1.